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Notes

On a trigonometric inequality of Turán

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Abstract

We prove the following theorem: Let $m \geq 2$ be a given integer and let a, b, c be real numbers. The inequalities

$$\sum_{k=1}^{n} \binom{m+n-k}{m} \sin(kx) > ax^2 + bx + c > 0$$

hold for all integers $n \geq 2$ and real numbers $x \in (0, \pi)$ if and only if

$$-\frac{m-1}{\pi} \leq a < 0, \qquad b = -a\pi, \qquad c = 0.$$

This refines a result due to Turán.

Keywords: Trigonometric polynomials; Inequalities; Sine integral

A classical result in the theory of trigonometric polynomials is the inequality

$$\sum_{k=1}^{n} \frac{\sin(kx)}{k} > 0 \quad (n \geq 1;\ 0 < x < \pi). \tag{1}$$

The validity of (1) was conjectured by Fejér in 1910. One year later, Jackson [4] published the first proof. The Fejér–Jackson inequality has attracted the attention of many mathematicians,

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who presented not only new proofs but also interesting extensions and numerous inequalities for other trigonometric sums. We refer to [1, Chapter 7], [2,3], [5, Chapter 4] and the references given therein.

In this note we are concerned with a class of sine polynomials studied by Turán [8] in 1935. He proved by induction the following remarkable inequality.

Proposition. *If* $m, n \geq 1$, *and* $x \in (0, \pi)$, *then*

$$\sum_{k=1}^{n} \binom{m+n-k}{m} \sin(kx) > 0. \tag{2}$$

Six years after the publication of Turán's paper, Szegö [7] provided a new proof of (2) for the special case $m = 2$ and used this result to show that the power series $F(z) = \sum_{k=0}^{\infty} a_k z^k$ is regular and univalent for $|z| < 1$ provided that the sequence $\{a_k\}$ is monotonic of order 3, that is, $\sum_{j=0}^{\nu} (-1)^j \binom{\nu}{j} a_{k+j} \geq 0$ for $\nu = 0, 1, 2, 3;\ k \geq 0$.

We ask: is it possible to approximate the sine polynomial given in (2) from below by an algebraic polynomial? More precisely, we are looking for an algebraic polynomial P of smallest degree such that

$$\sum_{k=1}^{n} \binom{m+n-k}{m} \sin(kx) > P(x) > 0 \tag{3}$$

is valid for all $n \geq 2$ and $x \in (0, \pi)$. Here, m is a fixed integer with $m \geq 2$.

There is no such polynomial of degree 1. We assume that $P(x) = ax + b$ $(a \neq 0)$. For any $n \geq 1$, the left-hand member of (3) has the limit 0 as $x \to 0$ or $x \to \pi$. Hence, we must have $b = 0$ and $a = 0$, that is $P \equiv 0$, contradicting the requirement that P is positive on $(0, \pi)$.

It is our aim to determine all polynomials P of degree 2 such that (3) is valid for all $n \geq 2$ and $x \in (0, \pi)$.

Theorem. *Let* $m \geq 2$ *be a given integer and let* a, b, c *be real numbers. The inequalities*

$$\sum_{k=1}^{n} \binom{m+n-k}{m} \sin(kx) > ax^2 + bx + c > 0 \tag{4}$$

hold for all integers $n \geq 2$ *and real numbers* $x \in (0, \pi)$ *if and only if*

$$-\frac{m-1}{\pi} \leq a < 0, \qquad b = -a\pi, \qquad c = 0. \tag{5}$$

Proof. First, we suppose that (4) is valid for $n = 2$ and $x \in (0, \pi)$:

$$\sin(x)[m + 1 + 2\cos(x)] > ax^2 + bx + c > 0.$$

We let x tend to 0 and π, respectively. This yields $c = 0$ and $b = -a\pi$. Thus,

$$\frac{\sin(x)}{\pi - x}[m + 1 + 2\cos(x)] > -ax > 0.$$

If x tends to π, then $\sin(x)/(\pi - x) \to 1$. It follows that $m - 1 \geq -a\pi > 0$, that is,

$$-\frac{m-1}{\pi} \leq a < 0.$$

Next, we show that conversely if (5) holds, then (4) is valid for all $m, n \geq 2$ and $x \in (0, \pi)$. It suffices to prove that

$$\sum_{k=1}^{n} \binom{m+n-k}{m} \sin(kx) > \frac{m-1}{\pi} x(\pi - x). \tag{6}$$

For $m, n \geq 1$ we denote the sum on the left-hand side of (6) by $S_{m,n}(x)$. In what follows, we prove for $m, n \geq 2$ the following chain of inequalities, which refines (6):

$$S_{m,n}(x) > \binom{m+n-3}{m-2} \sin(x) \geq (m-1)\sin(x) > \frac{m-1}{\pi} x(\pi - x). \tag{7}$$

To establish the first inequality in (7) for $m, n \geq 2$ we perform induction on m. Using the binomial formula

$$\binom{N}{m+1} - \binom{N-1}{m+1} = \binom{N-1}{m}$$

we get for $m \geq 1, k \geq 2$:

$$S_{m+1,k}(x) - S_{m+1,k-1}(x) = S_{m,k}(x). \tag{8}$$

The case $m = 1$ of (2) is due to Lukács. It states that

$$S_{1,n}(x) = \sum_{k=1}^{n} (n-k+1)\sin(kx) > 0 \quad (n \geq 1;\ 0 < x < \pi),$$

see [3, p. 8]. Applying (8) with $m = 1$ and $k = n$ this leads to

$$S_{2,n}(x) - S_{2,n-1}(x) = S_{1,n}(x) > 0 \quad (n \geq 2).$$

It follows that

$$S_{2,n}(x) > S_{2,n-1}(x) > \cdots > S_{2,2}(x) > S_{2,1}(x) = \sin(x).$$

This settles the case $m = 2$. For $m \geq 1, n \geq 2$ we have

$$S_{m+1,n}(x) = S_{m+1,1}(x) + \sum_{k=2}^{n} \big[S_{m+1,k}(x) - S_{m+1,k-1}(x)\big].$$

Invoking (8) we thus obtain

$$S_{m+1,n}(x) = \sin(x) + \sum_{k=2}^{n} S_{m,k}(x). \tag{9}$$

The identity (9) and the induction hypothesis yield

$$\begin{aligned}\frac{S_{m+1,n}(x)}{\sin(x)} &> 1 + \sum_{k=2}^{n} \binom{m+k-3}{m-2} \\ &= 1 + \sum_{k=2}^{n} \left[\binom{m+k-2}{m-1} - \binom{m+k-3}{m-1}\right] = \binom{m+n-2}{m-1}.\end{aligned}$$

This establishes the first inequality in (7).

The second inequality in (7) holds because

$$\binom{m+n-3}{m-2} = (m-1)\prod_{k=2}^{n-1}\left(1+\frac{m-2}{k}\right) \geq m-1,$$

the product over k being equal to 1 for $n=2$.

Since

$$\sin(x) \geq \frac{x(\pi-x)}{\pi}\left(1+\frac{x(\pi-x)}{\pi^2+x^2}\right) > \frac{x(\pi-x)}{\pi}, \tag{10}$$

we conclude that the third inequality in (7) is valid. The former inequality in (10) is due to Redheffer [6]. The theorem is proved. □

Remarks. (i) If $m=1$, then there are no real numbers a, b, c such that (4) holds for all $n \geq 2$ and $x \in (0,\pi)$. Otherwise, if $x \to 0$ and $x \to \pi$, then $c=0$ and $b=-a\pi$, respectively. Hence,

$$\sum_{k=1}^{n}(n-k+1)\frac{\sin(kx)}{\pi-x} > -ax > 0.$$

We let x tend to π and get

$$\frac{n+1}{4}[1+(-1)^{n+1}] = \sum_{k=1}^{n}(-1)^{k-1}k(n+1-k) \geq -a\pi \geq 0. \tag{11}$$

The expression on the left-hand side of (11) is equal to 0, if n is even. This leads to $a=b=c=0$. A contradiction.

(ii) If $n=1$, then (4) reads

$$\sin(x) > ax^2+bx+c > 0. \tag{12}$$

We let x tend to 0 and π, respectively, and apply the third inequality in (7). This reveals that (12) is true for all $x \in (0,\pi)$ if and only if $-1/\pi \leq a < 0$, $b=-a\pi$, $c=0$.

(iii) From (7) we get

$$\sum_{k=1}^{n}\binom{m+n-k}{m}\sin(kx) > A_{m,n}\sin(x)$$

with $A_{m,n} = \binom{m+n-3}{m-2}$. If $n=2$, then $A_{m,2}=m-1$ is the best possible factor. One of the referees raised the question: what is the best $B_m = A_{m,n}$, if it depends on m only, and $n \geq N$, where N is fixed?

(iv) If we replace in (6) x by $\pi - x$, then we obtain an inequality for an alternating sine sum:

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{m+n-k}{m}\sin(kx) > \frac{m-1}{\pi}x(\pi-x)$$
$$(m,n\geq 2;\ 0<x<\pi). \tag{13}$$

From (6) and (13) we conclude that (6) remains valid if we sum only over the terms with odd k:

$$\sum_{\substack{k=1\\ k\ odd}}^{n}\binom{m+n-k}{m}\sin(kx) > \frac{m-1}{\pi}x(\pi-x) \quad (m,n\geq 2;\ 0<x<\pi).$$

(v) The following chain of inequalities involves the two sine polynomials given in (1) and (2):

$$\frac{\pi}{x}\sum_{k=1}^{n}\binom{m+n-k}{m}\sin(kx) > \pi - x > \sum_{k=1}^{n}\frac{\sin(kx)}{k} > 0$$
$$(m \geq 2,\ n \geq 1;\ 0 < x < \pi). \tag{14}$$

The first inequality in (14) follows from (7), whereas the second one, which offers an upper bound for the Fejér–Jackson polynomial, was proved by Turán [9] in 1938.

(vi) If, in (6), we set $x = a + b$ and $x = a - b$, respectively, and sum the two terms on both sides of the inequalities, then we get an inequality providing a positive lower bound for a trigonometric sum involving the sine and cosine functions:

$$\sum_{k=1}^{n}\binom{m+n-k}{m}\sin(ka)\cos(kb) > (m-1)\left(a - \frac{a^2+b^2}{\pi}\right)$$
$$(m, n \geq 2;\ 0 < a + b < \pi,\ 0 < a - b < \pi). \tag{15}$$

This result contains (6) as special case. Indeed, if $b = 0$, then (15) reduces to (6).

(vii) If we divide both sides of (6) by x and integrate from 0 to a, then we obtain an inequality involving the sine integral $\mathrm{Si}(x) = \int_0^x \sin(t)dt/t$:

$$\sum_{k=1}^{n}\binom{m+n-k}{m}\mathrm{Si}(ka) > \frac{m-1}{\pi}a\left(\pi - \frac{1}{2}a\right) \quad (m, n \geq 2;\ 0 < a \leq \pi).$$

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References

[1] G.E. Andrews, R. Askey, R. Roy, Special Functions, Camb. Univ. Press, Cambridge, 1999.
[2] R. Askey, Orthogonal Polynomials and Special Functions, in: Reg. Conf. Ser. Appl. Math., vol. 21, SIAM, Philadelphia, PA, 1975.
[3] R. Askey, G. Gasper, Inequalities for polynomials, in: A. Baernstein II, D. Drasin, P. Duren, A. Marden (Eds.), The Bieberbach Conjecture, in: Math. Surveys and Monographs, No. 21, Amer. Math. Soc., Providence, RI, 1986, pp. 7–32.
[4] D. Jackson, Über eine trigonometrische Summe, Rend. Circ. Mat. Palermo 32 (1911) 257–262.
[5] G.V. Milovanović, D.S. Mitrinović, Th.M. Rassias, Topics in Polynomials: Extremal Problems, Inequalities, Zeros, World Sci. Publ., Singapore, 1994.
[6] R. Redheffer, Problem 5642, Amer. Math. Monthly 76 (1969) 422.
[7] G. Szegö, Power series with multiply monotonic sequences of coefficients, Duke Math. J. 8 (1941) 559–564.
[8] P. Turán, Über die arithmetischen Mittel der Fourierreihe, J. Lond. Math. Soc. 10 (1935) 277–280.
[9] P. Turán, Über die Partialsummen der Fourierreihe, J. Lond. Math. Soc. 13 (1938) 278–282.